Even Pace Is Not Even Effort

A Grade-Aware Model to Help Run Faster on Hilly Courses

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Abstract

Hills have a habit of ruining neat pace plans. A speed that feels comfortable on flat ground can feel like a sprint uphill and a braking drill downhill. I began with a simple question: if a runner has the same overall tolerance for hard effort, where should the faster and slower parts of a race go? A total-energy limit sounds useful, but it does not solve the problem here. Because the model treats metabolic cost per meter as nearly independent of speed, speed cancels from the route-energy equation. Every positive pace profile uses the same modeled net energy on the same route. I therefore compare strategies with a dimensionless intensity load that penalizes high power more strongly. Using the grade-cost equation measured by Minetti et al., I show that the fastest unconstrained strategy holds modeled power constant. Jensen's inequality also proves that this strategy cannot be slower than constant speed at the same modeled load. I test the result on a nominal 10-km horizontal profile with 122 m of climbing and grades from -5.34% to +5.34%. For a runner calibrated to 45:00 on flat ground, constant pace finishes in 47:15, while constant effort finishes in 46:05, saving 70.7 s, or 2.49%. A 4.5 m/s downhill cap changes the result by only 1.16 s. The model is a useful benchmark, not a magic race recipe: it leaves out fatigue history, wind, turns, surface conditions, and the fact that human legs may object to a very fast descent.

1 Background

1.1 Why Hills Turn Pacing Into an Optimization Problem

On a track, even pace is wonderfully boring. On a hilly course, the road refuses to cooperate. Holding the same speed uphill requires much more metabolic power. Holding it downhill may feel easier aerobically, but the runner still has to brake, balance, and absorb impact. The stopwatch can show the same pace while the legs are doing two very different jobs.

Experienced runners already adjust. In a study of eight runners on an outdoor hilly course, Townshend, Worringham, and Stewart found that they ran 23% slower uphill and 13.8% faster downhill than on level sections. The effect of a hill also continued after the grade returned to level [3]. That makes intuitive sense, but watching what runners naturally do is not the same as proving what is fastest.

My mathematical question is: for a fixed route and a fixed measure of tolerable exertion, how should speed change with position to minimize finish time? To answer it fairly, I need to define the route carefully, use a measured running-cost equation, and compare every strategy at the same modeled load. I also need to remember that a clean equation is not automatically a coaching command.

1.2 What Is Known About Grade and Running Cost

Minetti et al. measured the metabolic cost of running over grades ranging from steep downhill to steep uphill [1]. Cost per meter rose quickly uphill, fell on moderate descents, and then rose again on very steep descents. That last turn matters. Downhill running is not free, and a simple straight-line formula can eventually predict an impossible negative energy cost.

Adding only the gravitational work of lifting the body is also too simple. Hoogkamer, Taboga, and Kram showed that braking and propulsion change as the slope changes [2]. I therefore use an experimental grade-cost curve instead of inventing a convenient hill penalty.

Metabolic cost is only part of the story. Gottschall and Kram measured much larger impact and braking forces on a 9-degree descent than on level ground [5]. In other words, a downhill can be cheap for the lungs and expensive for the legs. That is why I later test a speed cap instead of letting the mathematical runner fly downhill without consequences.

1.3 The Modeling Gap

This is not the first attempt to optimize trail-running pace. Jaszczak and Plociniczak built a much larger model with terrain, fatigue, and nutrition, then tested it on real mountain-race profiles [4]. My goal is smaller. I want a model simple enough to solve by hand, so it is obvious which assumption causes which result.

The main trap is the constraint. If energy cost per meter does not depend on speed, then the total route energy is fixed. Calling it an energy budget does not suddenly make one pacing plan better than another. The model needs a second quantity that cares about how hard the runner works at each moment.

1.4 Research Question and Contributions

I use that idea to test whether a small, internally consistent model can explain the familiar advice to run hills by effort instead of pace. The project does five main things:

- separates horizontal distance, surface distance, grade, and actual running speed;
- uses a measured, positive grade-cost equation instead of a made-up linear penalty;
- shows why route energy alone cannot choose a pace plan;
- derives a constant-power optimum from a nonlinear intensity-load model;
- proves that equal-load constant speed cannot be faster; and
- tests the result with speed bounds, sensitivity checks, mesh checks, and noisy elevation data.

2 Model Concept

2.1 Route Geometry and True Distance

Let x be horizontal course distance and $z(x)$ be elevation. Local grade is the rise-over-run ratio

$$g(x) = \mathrm{d}z/\mathrm{d}x.$$

The runner travels along the sloped surface, so the distance covered is not simply dx. It is

$$\mathrm{d}s = \sqrt{(1 + g(x)^2)}\ \mathrm{d}x.$$

I describe the route using surface arclength s on $[0, L]$. The speed $v(s) = \mathrm{d}s/\mathrm{d}t$ is the runner's actual speed along the road. Keeping horizontal and surface distance separate avoids a small-looking geometry mistake that would spread through every later equation.

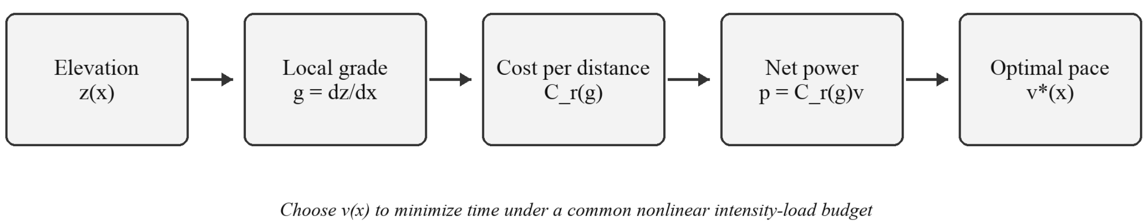

Figure 1: Modeling pipeline. Elevation determines grade; grade determines cost per surface distance; cost and speed determine modeled net metabolic power; the pacing problem selects a speed profile under a common modeled load.

2.2 Grade-Dependent Cost of Transport

Let $C_r(g)$ be net metabolic energy per unit body mass and surface distance, in J kg^{-1} m^{-1}. I use the fifth-order equation fitted by Minetti et al. [1]:

$$C_r(g) = 155.4g^5 - 30.4g^4 - 43.3g^3 + 46.3g^2 + 19.5g + 3.6.$$

Here g is a decimal fraction, so 0.05 means a 5% grade. I use the equation only over the moderate grades on my synthetic route. It comes from steady treadmill running, which means it knows nothing about wind, sharp turns, loose gravel, heat, altitude, or tired legs late in a race.

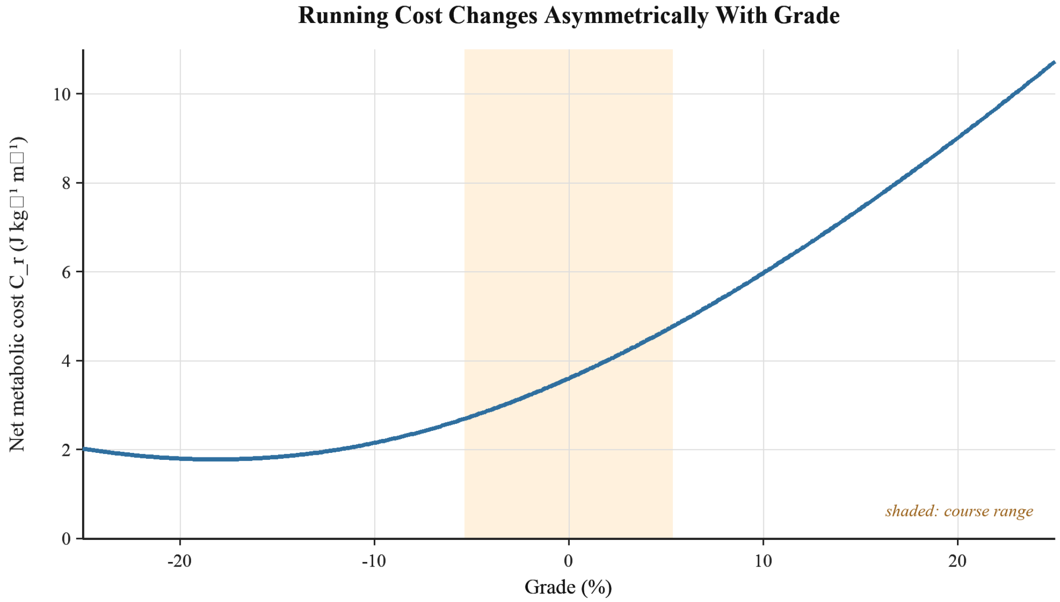

Figure 2: Minetti running-cost polynomial over grades from -25% to +25%. The shaded band marks the -5.34% to +5.34% range used in the controlled course. The curve is asymmetric and remains positive; moderate descent is cheaper than level running, while steep descent becomes more expensive.

2.3 Why a Total-Energy Constraint Is Insufficient

Modeled specific net metabolic power is cost per meter multiplied by speed:

$$p(s) = C_r(g(s))\, v(s),$$

where p has units W kg^{-1}. The total modeled net energy per kilogram is

$$E/m = \int_0^T p(t)\, \mathrm{d}t = \int_0^L C_r(g(s))\, \mathrm{d}s.$$

Because $\mathrm{d}t = \mathrm{d}s/v$, speed cancels. That is the algebraic plot twist of the paper. On a fixed route, the model assigns the same total energy to constant pace, constant effort, and every other positive speed profile. An energy ceiling might say whether the route is possible, but it cannot choose the fastest schedule. I need another quantity that responds to high intensity.

2.4 A Nonlinear Intensity-Load Surrogate

To compare different intensity patterns, let T_0 be the duration of a reference performance. I define the dimensionless load

$$F_q[v] = (1/T_0) \int_0^T [p(t)/p_{\mathrm{ref}}]^q\, \mathrm{d}t, \quad q > 1,$$

Here p_{ref} is a runner-specific reference power, and q controls how strongly the model dislikes power spikes. Changing variables from time to distance gives

$$F_q[v] = [1/(T_0 p_{\mathrm{ref}}^q)] \int_0^L C_r(g(s))^q\, v(s)^{q-1}\, \mathrm{d}s.$$

This is a mathematical stand-in for intensity load. It does not directly measure glycogen, lactate, heart-rate load, critical speed, or injury risk. Its one job is to make concentrated high

power more costly than moderate power held for the same time. That creates a fair optimization, but q would have to be fitted to real athlete data before anyone should use the output as a race plan.
The minimum-time problem is

$$\text{minimize } \; T[v] = \int_0^L ds/v(s) \; \text{ subject to } \; F_q[v] \le 1.$$

2.5 Illustrative Speed Bounds

The basic model can recommend a very fast downhill speed whenever metabolic cost is low. For a more realistic comparison, I also impose

$$v_{\min}(s) \le v(s) \le v_{\max}(s).$$

These are example feasibility limits, not universal safety rules. I use 4.5 m/s as the upper bound and 2.5 m/s as the lower bound only to see how clipping changes the answer. A real limit should depend on the runner, grade, turns, surface, visibility, shoes, fatigue, and tolerance for downhill impact.

3 Methodology

3.1 Synthetic Course

I built a synthetic route so that every input can be reproduced exactly. Its horizontal length is L_x = 10,000 m, and elevation is

$$z(x) = 40[1 - \cos(2\pi x/L_x)] + 15[1 - \cos(6\pi x/L_x)] + 6[1 - \cos(10\pi x/L_x)] \text{ meters.}$$

The route climbs to one main summit and then descends to its starting elevation. The smaller cosine terms change the steepness along the way without creating extra climbs. Total gain is 122 m, and grade ranges from -0.05344 to +0.05344. After the arclength correction, the runner travels 10,004.46 m, so '10 km' refers to horizontal distance rather than exact surface distance. The smooth profile is deliberate: I wanted to test the pacing logic without hiding it behind rocks, turns, or messy map data.

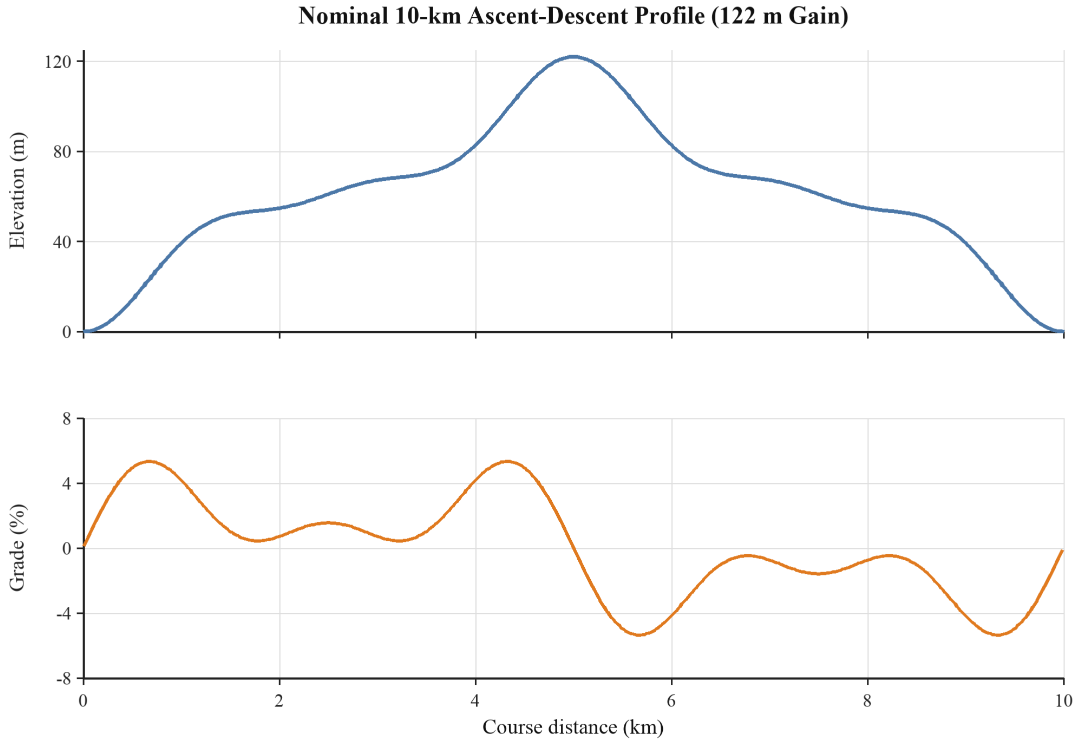

Figure 3: Elevation and local grade for the nominal 10-km horizontal profile. The route has 122 m of gain, returns to its starting elevation, and remains within approximately ±5.34% grade.

3.2 Discretization

For the computation, I divided the horizontal route into 500 segments of 20 m. At each midpoint, I evaluated the exact grade, calculated surface length as $\Delta s_i = \Delta x\sqrt{(1 + g_i^2)}$, and used the Minetti equation to find C_i. If segment i is run at speed v_i, the discrete time and load are

$$T_N = \Sigma_{i=1}^{N} \Delta s_i/v_i,$$

$$F_{q,N} = \Sigma_{i=1}^{N} \Delta s_i C_i^q v_i^{q-1}/(T_0 p_{\mathrm{ref}}^q).$$

3.3 Runner Calibration and Equal-Load Comparison

I calibrate the reference runner to a flat 10 km time of T_0 = 2700 s, or 45:00. That equals 3.7037 m/s and 4:30 min/km. Since $C_r(0)$ = 3.6 J kg^{-1} m^{-1}, the reference specific net power is p_{ref} = 13.333 W kg^{-1}. The main calculation uses $q = 2$ as an example; I did not fit this value to athlete data. The flat reference run has $p = p_{\mathrm{ref}}$ for T_0, so $F_2 = 1$. In the code, the same normalization is $K = T_0 p_{\mathrm{ref}}^2$ = 480,000 W^2 s kg^{-2}.

I compare three strategies at exactly the same load:

- **Constant pace:** one surface speed, chosen so that $F_2 = 1$.
- **Constant effort:** the unconstrained optimum, which keeps modeled specific net power constant.
- **Speed-bounded optimum:** the same optimization with example limits of 2.5 and 4.5 m/s.

3.4 Numerical Solution of the Bounded Case

Without speed bounds, the optimum uses the same power everywhere. With lower and upper limits, the Karush-Kuhn-Tucker condition keeps a common interior power p_{bar} and clips it only where that power would require an illegal speed:

$$p_i = \mathrm{clip}(p_{\mathrm{bar}}, C_i v_{\mathrm{min}}, C_i v_{\mathrm{max}}).$$

If the bounds allow at least one profile with the target load, bisection finds p_{bar} without depending on a lucky initial guess. Here the 4.5 m/s upper limit activates on low-cost descents, while the 2.5 m/s lower limit never does. The model still ignores acceleration and braking limits between segments. In the continuous formula, clipping keeps speed continuous but can create derivative kinks where the cap begins; the program represents that profile with 20 m segment values.

3.5 Sensitivity Analysis

I scale the course amplitude from 0 to 1.5 times its original value. Total gain changes from 0 to 183 m, while the horizontal waveform and 10,000 m horizontal length stay the same; surface length changes slightly through the arclength correction. At each scale, I recompute constant pace and constant effort for q = 1.5, 2, and 3. This tests how the size of the predicted advantage changes with the exponent and with cost variability. It does not claim that elevation gain alone controls pacing on every course.

3.6 Elevation-Noise Experiment

Grade comes from differentiating elevation, and differentiation is unforgiving: a small height error can become a large grade error. Sport-watch research shows that reported climbing depends on the device and its filtering [6]. To see the effect, I add one seeded sample of independent Gaussian noise with standard deviation 0.75 m to elevations spaced every 20 m. I calculate grade from the raw data and again after a reflected 31-point Hann window spanning 600 m between its first and last samples. This is a demonstration, not a claim that 600 m is the perfect filter for every route.

3.7 Verification Checks

I required the program to pass four basic checks: a flat route had to return 45:00, every strategy had to use the same normalized load, unconstrained optimal power had to be constant to numerical precision, and shrinking segment size from 100 m to 10 m could not materially change the answer. These checks test my algebra and code. They do not prove that the fatigue model matches a real runner.

4 Results and Verification

4.1 Analytical Constant-Power Optimum

The unconstrained problem can be solved before doing any numerical optimization. Its Lagrangian integrand is

$$1/v + \lambda C_r^{\,q} v^{q-1}/(T_0 p_{\mathrm{ref}}^{\,q}).$$

Differentiating with respect to v gives

$$-1/v^2 + \lambda(q - 1)C_r^q v^{q-2}/(T_0 p_{ref}^q) = 0.$$

After multiplying by v^2, all factors except $(C_r v)^q$ are constant. Therefore the interior solution satisfies

$$p(s) = C_r(g(s))v(s) = p^*.$$

The load constraint must be active. If $F_q < 1$, multiplying every speed by $\alpha > 1$ divides time by α and multiplies load by α^{q-1}. Increasing α until the load reaches one gives a faster feasible run. Inside this model, 'run by effort' is a result, not an assumption. If $A = \int_0^L C_r \, ds$, the active constraint gives

$$p^* = [T_0 p_{ref}^q/A]^{1/(q-1)}, \quad v^*(s) = p^*/C_r(g(s)).$$

To prove global optimality, define $d\mu = C_r ds/A$ and $X = p^{q-1}$. At fixed active load, $E_\mu[X]$ is fixed, while $T = AE_\mu[X^{-1/(q-1)}]$. For positive cost, positive speed, and $q > 1$, this last power is strictly convex. Jensen's inequality says time is smallest only when X, and therefore p, is constant almost everywhere. The global optimum is unique.

4.2 Why Constant Speed Cannot Be Faster

Now let $B = \int_0^L C_r^q \, ds$, where L is total surface length. A constant speed using the same load is

$$v_{const} = [T_0 p_{ref}^q/B]^{1/(q-1)}.$$

The ratio of its finish time to the optimum is

$$T_{const}/T^* = [L^{q-1}B/A^q]^{1/(q-1)} \geq 1.$$

On surface distance, Jensen's inequality gives $\mathrm{mean}(C_r^q) \geq \mathrm{mean}(C_r)^q$, exactly the inequality above. Equality occurs only when cost is constant almost everywhere. At equal modeled load, constant speed can tie the optimum on uniform terrain, but it cannot beat it when cost varies.

4.3 Controlled Nominal 10-km Case

On the synthetic route, every strategy uses 36,429.49 J kg^{-1} of modeled net locomotion energy, as the cancellation argument predicts. The quadratic load, not route energy, creates the time differences in Table 1.

Strategy	Finish time	Change vs. constant pace	Mean power (W kg^{-1})	Speed range (m/s)
Constant pace	47:15	reference	12.848	3.528
Constant effort	46:05	-70.7 s (-2.49%)	13.176	2.764-4.886
Speed-bounded optimum	46:06	-69.5 s (-2.45%)	13.171	2.782-4.500

Table 1: Equal-load pacing results for the nominal 10-km profile; negative changes indicate faster completion.

Constant pace settles at 3.528 m/s, or about 4:43.5 min/km. As grade changes, its modeled specific net power swings from 9.514 to 16.821 W kg^{-1}. The unconstrained optimum holds power at 13.176 W kg^{-1} and lets speed range from 2.764 to 4.886 m/s. It loses time on the opening climb, then gains more on the descent. The final result is 46:05 instead of 47:15: 70.7 s faster, or 2.49%, in this synthetic example.

4.4 Pace and Power Profiles

The upper panel makes the main idea visible. Constant pace looks tidy, but maximum modeled power is about 77% above minimum power. Constant modeled power looks uneven on a watch because speed changes with grade. The bounded solution stays close to constant power until the 4.5 m/s downhill cap activates.

Here the cap adds only 1.16 s, producing a 46:06 finish and a 69.5 s advantage over constant pace. That small penalty belongs to this route and this chosen cap. Longer low-cost descents, a lower limit, or technical terrain could make the bound much more important. It is a demonstration, not a universal safe speed.

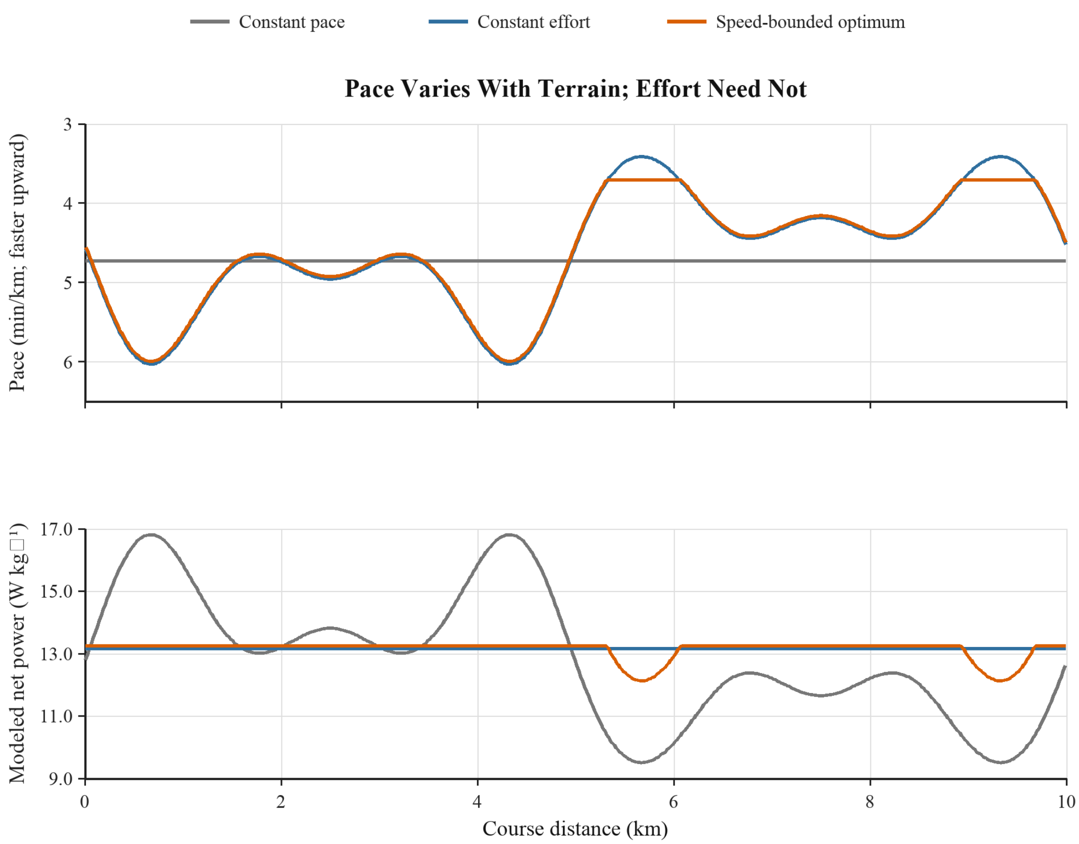

Figure 4: Pace (top) and modeled specific net metabolic power (bottom) for the three equal-load strategies. Constant pace produces large power spikes on climbs. Constant effort varies pace inversely with cost. The speed-bounded profile clips speed at 4.5 m/s on the lowest-cost descents and reallocates a small amount of load elsewhere.

4.5 Cumulative Time Tradeoff

At intermediate checkpoints, the optimum can look surprisingly bad. Near 5 km it is about 145 s behind constant pace because the first half is mostly uphill. The descent reverses the gap, and the optimized profile moves ahead during the ninth kilometer before finishing about 70 s faster. One split does not reveal whether the full-race strategy is efficient. If the same cost-and-distance segments were rearranged, the intermediate gaps would change, but the final analytical times would not. The model has no acceleration cost, fatigue history, or memory of what happened earlier.

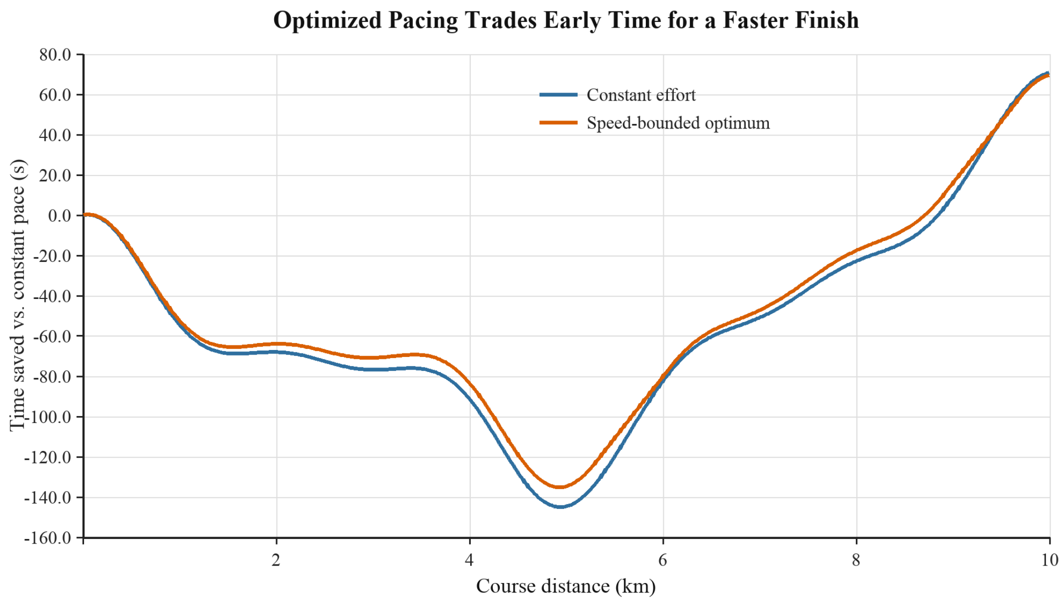

Figure 5: Cumulative time saved relative to equal-load constant pace. Negative values mean the terrain-aware strategy is temporarily behind. The optimized runners lose time on the opening climbs, then recover it as the course descends, finishing approximately 70 s ahead.

4.6 Sensitivity to Terrain and Load Exponent

For the base profile with 122 m of gain, constant effort beats constant pace by 1.88% when q = 1.5, 2.49% when q = 2, and 3.68% when q = 3. At 183 m of gain, the values become 4.03%, 5.28%, and 7.65%. Every curve starts at zero on flat ground, as the proof requires. A larger q punishes power spikes more strongly, so smoothing power becomes more valuable. These numbers belong to one scaled family of routes and should not be generalized to other courses.

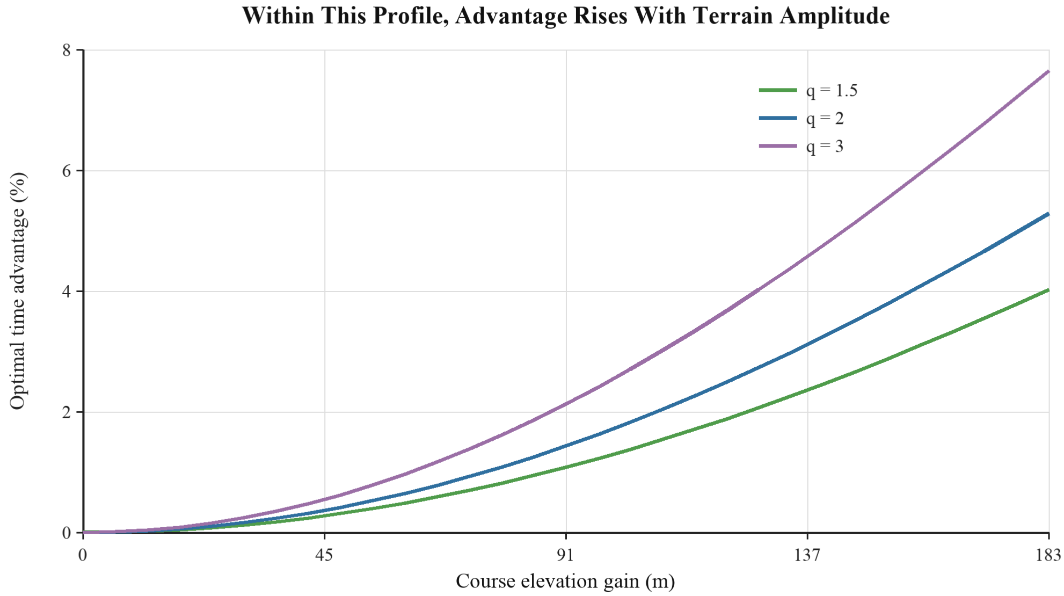

Figure 6: Within the scaled synthetic profile, percentage advantage of constant effort over equal-load constant speed as elevation amplitude increases. The benefit is zero on flat terrain and grows with both cost variability and the convexity exponent q.

4.7 Elevation Noise and Smoothing

In the single seeded noise example, differentiating the raw elevations gives a grade RMSE of 0.02593, or 2.59 percentage points. Smoothing lowers it to 0.001283, or 0.128 percentage points, about a 95% reduction. The catch is that a window spanning 600 m can flatten real short hills along with the noise. Preprocessing clearly matters, but one simulated noise sample does not identify the perfect filter or prove that finish-time predictions are robust.

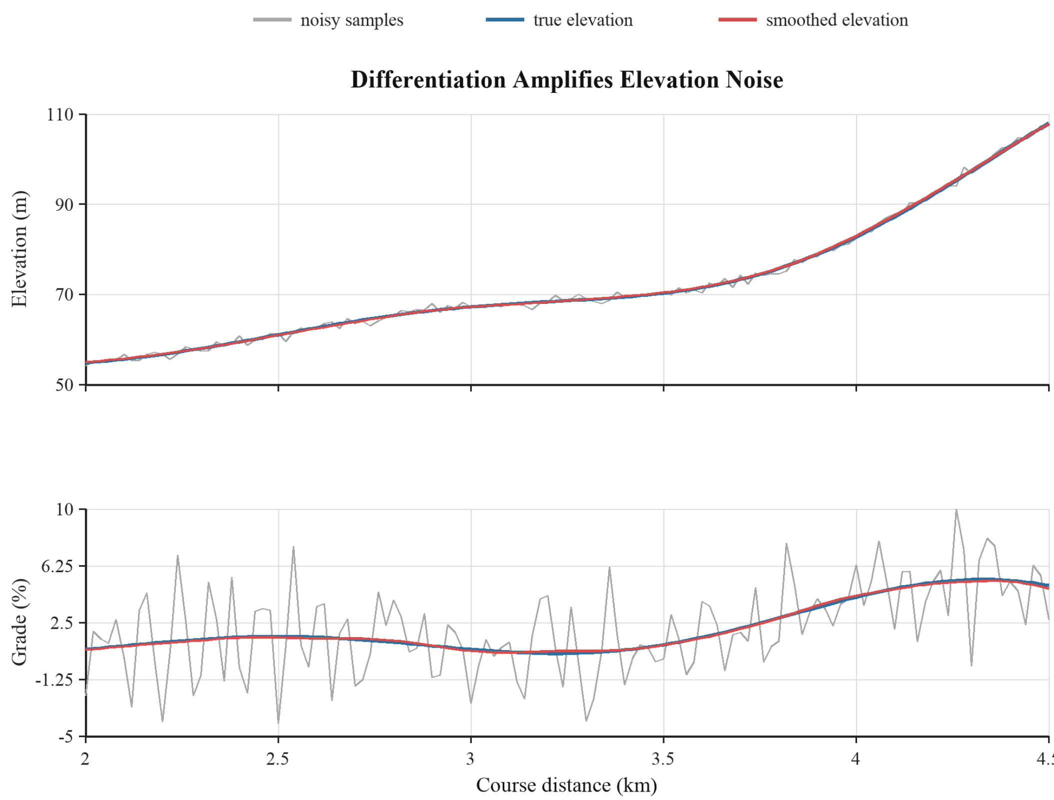

Figure 7: One seeded illustration of elevation noise. Independent 0.75 m errors are visually small in elevation (top) but become large local-grade oscillations after differentiation (bottom). A 31-point Hann filter greatly reduces the error while also smoothing real short-scale features.

4.8 Internal Verification

The flat-course test returns 2700.0 s, exactly matching the calibration. Across all three strategies, the largest relative load error is 3.64×10^{-16}, and the standard deviation of unconstrained optimal power is 3.55×10^{-15} W kg^{-1}. Both are floating-point roundoff. Segment lengths of 100, 50, 25, 20, and 10 m change finish time by less than 10^{-6} s. That unusually close agreement comes from the smooth periodic test route; real GPS elevation is rarely so polite.

5 Discussion and Future Work

5.1 What the Result Shows

The main result is simple, but it comes with conditions. If grade sets the metabolic cost of each meter, power equals cost times speed, and high power is penalized nonlinearly, then constant modeled net power is the unique fastest unconstrained strategy. Constant speed cannot be faster at the same modeled load. This is a mathematical result inside the model, not a universal rule for every runner.

The energy calculation also explains why an energy budget alone does not solve the pacing problem. All three strategies used the same modeled route energy because speed cancels from that integral. Their times differed only because the quadratic load penalized large power peaks. In short, the runner cannot fool the model by sprinting every climb and calling the total energy equal.

The predicted pattern agrees with the hill-running behavior observed by Townshend et al.: runners generally slowed uphill and accelerated downhill [3]. That agreement is encouraging, but it is not validation. The study does not show that those runners minimized F_q, and oxygen uptake does not respond instantly to changes in metabolic power. More complete work has modeled terrain together with fatigue, nutrition, and propulsion dynamics [4], while critical-speed methods estimate a sustainable threshold and a finite reserve from running tests [7]. This model is best viewed as a baseline that those richer models should recover under simplified conditions.

5.2 Limits That Matter

The unconstrained solution reached 4.886 m/s on the lowest-cost descent. Metabolically cheap does not mean mechanically free. Downhill running can produce larger impact and braking forces [5], so the 4.5 m/s cap used here is only an example, not safety advice. A useful field model would need a grade-dependent limit based on the runner, surface, turns, visibility, and downhill skill.

The physiological assumptions are also simplified. The exponent $q = 2$ was chosen for illustration rather than fitted to an athlete. The Minetti relation came from steady treadmill running [1], and the model leaves out acceleration, wind, heat, oxygen-uptake delay, fatigue history, recovery, and an end sprint. The test course is synthetic, so the predicted advantage should not be treated as a race forecast.

Course data add another source of uncertainty. Elevation errors become much larger after elevation is differentiated into grade, and sport-watch filtering can change the reported climbing total [6]. The smoothing experiment demonstrated this problem with one seeded noise sample; it did not identify a universal filter. The equations behave nicely because the course is smooth, while real GPS data are rougher.

5.3 A Useful Next Experiment

The next step should test the model on runners rather than add more decimal places. Each participant could complete flat and graded economy trials, several hard efforts to estimate sustainable power and reserve, and controlled downhill runs to establish a reasonable speed limit. Parameters would be fitted on training runs and then frozen before testing on held-out time trials.

Evaluation should include split error by grade, predicted versus measured effort, downhill loading, and uncertainty from elevation processing, not just final time. The model should also be compared with constant pace and a common grade-adjusted pacing rule. If it predicts unseen runs more accurately without exceeding measured limits, then the stopwatch has finally given the model a vote.

6 Conclusion

This study built a consistent mathematical benchmark for pacing over variable terrain. A fixed-route energy integral cannot choose a pace distribution because speed cancels. Under the proposed nonlinear intensity-load budget, constant modeled net power is the unique unconstrained optimum, and Jensen's inequality proves that equal-load constant speed cannot be faster when grade-dependent cost varies.

On the nominal 10-km horizontal profile, constant effort finished in 46:05 compared with 47:15 for constant pace, a modeled improvement of 70.7 seconds or 2.49%. Adding the illustrative downhill cap changed the result to 46:06. These numbers belong to one synthetic course and one assumed runner; they are not a universal split chart. The broader lesson is more useful: define effort clearly, compare strategies at equal load, and treat physiological, mechanical, and course-data limits as part of the optimization rather than as footnotes.

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Artificial intelligence tools were used solely to improve clarity, organization, and grammatical precision of the written manuscript. All research design, mathematical modeling, data analysis, code implementation, experimental validation, and scientific conclusions were developed and executed independently by the author.